

11(b). Weighting Estimators

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Why weighting?

- Matching has a couple of downsides:
 - Inefficient: it may throw away data.
 - Ineffective: crude tool so it may not be able to balance covariates.
- Matching is actually a special case of a weighting estimator:

$$\begin{aligned}\hat{\tau}_m &= \frac{1}{n_1} \sum_{i=1}^n D_i \left(Y_i - \frac{1}{M} \sum_{j \in \mathcal{J}_M(i)} Y_j \right) \\ &= \frac{1}{n_1} \sum_{i:D_i=1} Y_i - \frac{1}{n_0} \sum_{i:D_i=0} \underbrace{\left(\frac{n_0}{n_1} \frac{K_M(i)}{M} \right)}_{\text{weight}} Y_i\end{aligned}$$

- $K_M(i)$ is the number of times i is used as a match.
- Weighting estimators choose the weights directly to reduce imbalance.

Survey sampling

- Imagine we want to estimate the population mean $\bar{Y} = \frac{1}{n} \sum_{i=1}^n Y_i$
 - $R_i = 1$ if sampled, $R_i = 0$ if not.
 - We only observe Y_i for those with $R_i = 1$.
 - Inclusion probability varies by person: $\mathbb{P}(R_i = 1) = \pi_i$
- **Horvitz-Thompson estimator** is unbiased (treating Y_i as fixed):

$$\mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \frac{R_i Y_i}{\pi_i} \right] = \frac{1}{n} \sum_{i=1}^n \frac{\mathbb{E}[R_i] Y_i}{\pi_i} = \frac{1}{n} \sum_{i=1}^n \frac{\pi_i Y_i}{\pi_i} = \bar{Y}$$

- Key idea: reweight sample to be representative of population.

Horvitz-Thompson for treatment effects

- Applying HT potential outcomes: weight by inverse propensity score.

$$\widehat{ATE} = \widehat{\tau}_{ipw} = \frac{1}{n} \sum_{i=1}^n \left(\frac{D_i Y_i}{\widehat{\pi}(\mathbf{X}_i)} - \frac{(1 - D_i) Y_i}{1 - \widehat{\pi}(\mathbf{X}_i)} \right)$$

- Under no unmeasured confounders, $\mathbb{E}[\widehat{\tau}_{ipw}] \xrightarrow{P} \tau$ (consistent)
 - Would be unbiased if we knew the true propensity scores, $\pi(\mathbf{X}_i)$
- Similar expression for ATT:

$$\widehat{ATT} = \widehat{\tau}_{ipw,t} = \frac{1}{n_1} \sum_{i=1}^n \left(D_i Y_i - \frac{\widehat{\pi}(\mathbf{X}_i)(1 - D_i) Y_i}{1 - \widehat{\pi}(\mathbf{X}_i)} \right)$$

- Logic: upweight units with “rare” treatment values for their values of \mathbf{X}_i
 - A kind of “continuous” version of matching with replacement.

Normalized weights

- HT estimators with known weights are unbiased but can be inefficient.
 - Large weights can lead to highly variable estimates when (not) included.
- **Hajek estimator** normalizes the denominator so the weights sum to 1:

$$\hat{\mu}_h = \hat{\mathbb{E}}[Y_i] = \frac{\sum_{i=1}^n (R_i/\pi_i) Y_i}{\sum_{i=1}^n R_i/\pi_i}$$

- Hajek estimator for the ATE:

$$\hat{\tau}_h = \frac{\sum_{i=1}^n D_i Y_i / \hat{\pi}(\mathbf{x}_i)}{\sum_{i=1}^n D_i / \hat{\pi}(\mathbf{x}_i)} - \frac{\sum_{i=1}^n (1 - D_i) Y_i / (1 - \hat{\pi}(\mathbf{x}_i))}{\sum_{i=1}^n (1 - D_i) / (1 - \hat{\pi}(\mathbf{x}_i))}$$

- Practically, weighted least squares gives automatic normalization:

$$(\hat{\alpha}_{\text{wls}}, \hat{\tau}_{\text{wls}}) = \arg \min_{\alpha, \tau} \sum_{i=1}^n \left(\frac{D_i}{\hat{\pi}(\mathbf{x}_i)} + \frac{1 - D_i}{1 - \hat{\pi}(\mathbf{x}_i)} \right) (Y_i - \alpha - \tau D_i)^2$$

Variance

- If $\widehat{\pi}(\mathbf{X}_i)$ is estimated, how to estimate $\mathbb{V}[\widehat{\tau}_{ipw}]$ or $\mathbb{V}[\widehat{\tau}_h]$?
- First option: **bootstrap** (possibly with clustering if needed)
- Second option: **method of moments** (Newey and McFadden, 1994)
 - Treat this a joint estimation problem and use the delta method.
 - Moment conditions for the propensity score model with parameters θ :

$$\mathbb{E} \left[\underbrace{\left(\frac{D_i}{\pi_\theta(\mathbf{X}_i)} - \frac{1 - D_i}{1 - \pi_\theta(\mathbf{X}_i)} \right) \frac{\partial \pi_\theta(\mathbf{X}_i)}{\partial \theta}}_{\text{score for treatment model}} \right] = 0$$

- Moment conditions for weighting estimators:

$$\text{HT: } \mathbb{E} \left[\frac{D_i Y_i}{\pi_\theta(\mathbf{X}_i)} - \mathbb{E}[Y_i(1)] \right] = \mathbb{E} \left[\frac{(1 - D_i) Y_i}{1 - \pi_\theta(\mathbf{X}_i)} - \mathbb{E}[Y_i(0)] \right] = 0$$

$$\text{Hajek: } \mathbb{E} \left[\frac{D_i (Y_i - \mathbb{E}[Y_i(1)])}{\pi_\theta(\mathbf{X}_i)} \right] = \mathbb{E} \left[\frac{(1 - D_i) (Y_i - \mathbb{E}[Y_i(0)])}{1 - \pi_\theta(\mathbf{X}_i)} \right] = 0$$

- Replace with sample versions and use delta method to get asymptotic variance.

Estimated versus known pscores

```
ht.est <- function(y, d, w) {  
  n <- length(y)  
  (1/n) * sum((y * d * w) - (y * (1-d) * w))  
}  
n <- 200  
x <- rbinom(n, size = 1, prob = 0.5)  
dprobs <- 0.5*x + 0.4*(1-x)  
d <- rbinom(n, size = 1, prob = dprobs)  
y <- 5 * d - 10 * x + rnorm(n, sd = 5)  
  
true.w <- ifelse(d == 1, 1/dprobs, 1/(1-dprobs))  
pprobs <- predict(glm(d ~ x))  
est.w <- ifelse(d == 1, 1/pprobs, 1/(1 - pprobs))  
ht.est(y, d, est.w)
```

```
## [1] 5.22
```

```
ht.est(y, d, true.w)
```

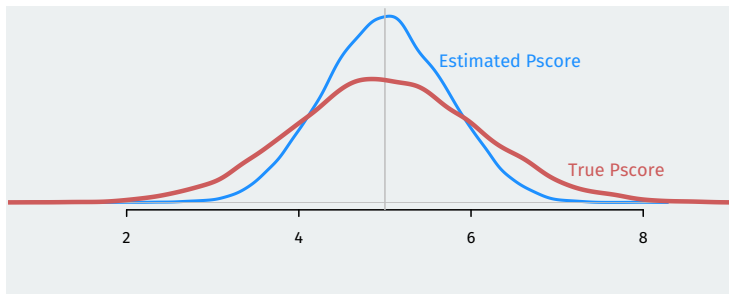
```
## [1] 5.56
```

Sampling distribution of the HT estimators

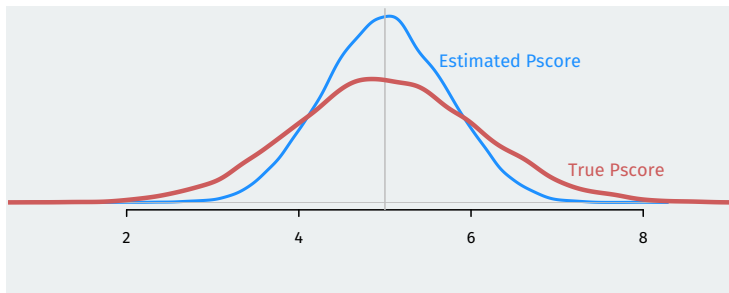
```
sims <- 10000
true.holder <- rep(NA, sims)
est.holder <- rep(NA, sims)
for (i in 1:sims) {
  x <- rbinom(n, size = 1, prob = 0.5)
  dprobs <- 0.5*x + 0.4*(1-x)
  d <- rbinom(n, size = 1, prob = dprobs)

  y <- 5 * d - 10 * x + rnorm(n, sd = 5)
  true.w <- ifelse(d == 1, 1/dprobs, 1/(1-dprobs))
  pprobs <- predict(glm(d ~ x))
  est.w <- ifelse(d == 1, 1/pprobs, 1/(1 - pprobs))
  est.holder[i] <- ht.est(y, d, est.w)
  true.holder[i] <- ht.est(y, d, true.w)
}
```

Sampling distribution of the HT estimators



Sampling distribution of the HT estimators



```
var(est.holder)
```

```
## [1] 0.506
```

```
var(true.holder)
```

```
## [1] 1.15
```

Why use estimated pscores?

- Why is the estimated propensity score more efficient than the true PS?
- **Removing chance variations** using $\hat{\pi}(\mathbf{X}_i)$ adjusts for any small imbalances that arise because of a finite sample.
- True PS only adjusts for the **expected** differences between samples.
- Only true if propensity score model is **correctly specified!!**

Augmented IPW estimator

- **Augmented** IPW estimator combines regression and weighting:

$$\widehat{\tau}_{\text{aipw}} = \frac{1}{n} \sum_{i=1}^n \left\{ \frac{D_i Y_i}{\widehat{\pi}(\mathbf{X}_i)} - \frac{(1 - D_i) Y_i}{1 - \widehat{\pi}(\mathbf{X}_i)} - \left(\frac{D_i - \widehat{\pi}(\mathbf{X}_i)}{\widehat{\pi}(\mathbf{X}_i)} \widehat{\mu}_1(\mathbf{X}_i) - \frac{D_i - \widehat{\pi}(\mathbf{X}_i)}{1 - \widehat{\pi}(\mathbf{X}_i)} \widehat{\mu}_0(\mathbf{X}_i) \right) \right\}$$

Augmented IPW estimator

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Augmented IPW estimator

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- **Double robustness:** consistent if either $\widehat{\pi}(\mathbf{X}_i)$ or $\widehat{\mu}_d(\mathbf{X}_i)$ are consistent.
 - Can allow each model to be more flexible without hurting asymptotics.
- **Efficient:** lowest asymptotic variance among consistent estimators when PS model is correct.

Problems with weighting

- Propensity score balances covariates across **repeated samples**
 - Relies on the law of large numbers (not optimized for small samples)
- Difficult to model propensity score for covariate balance.
- Highly variable/unstable weights \leadsto high variance estimators.
 - When there is a lack of overlap so $\pi(\mathbf{X}_i)$ is close to 0 or 1.
 - **Windsorizing**: trim weights beyond 5th and 9th percentile

Covariate balancing propensity scores

- How to help with these issue: prioritize balance in estimation of $\pi(\mathbf{X}_i)$
 - One approach: **covariate balancing propensity scores** (Imai and Ratkovic)

- Usual maximum likelihood estimation solves a moment condition:

$$\mathbb{E} \left[\underbrace{\left(\frac{D_i}{\pi_\theta(\mathbf{X}_i)} - \frac{1 - D_i}{1 - \pi_\theta(\mathbf{X}_i)} \right) \frac{\partial \pi_\theta(\mathbf{X}_i)}{\partial \theta}}_{\text{score}} \right] = 0$$

- CBPS adds balancing conditions:

$$\mathbb{E} \left[\left(\frac{D_i}{\pi_\theta(\mathbf{X}_i)} - \frac{1 - D_i}{1 - \pi_\theta(\mathbf{X}_i)} \right) f(\mathbf{X}_i) \right] = 0$$

- If $f(\mathbf{X}_i) = \mathbf{X}_i$, then condition says the means must be balanced.
- If $f(\mathbf{X}_i) = \mathbf{X}_i^2$, then balances the second moments.
- Estimation by generalized method of moments (GMM).
 - More moment conditions than parameters \leadsto can't satisfy all conditions.
 - Find parameters that get closest to satisfying sample moment conditions

Balancing weights

- Alternative: find weights that directly balance covariates.
- **Stable balancing weights** (Zubizarreta, 2015) one recent example.
 - Rearrange data so first m units are controls, rest are treated.
- Solve the following convex quadratic programming problem:

$$\min_{w_1, \dots, w_m} \sum_{i=1}^m (w_i - \bar{w})^2$$

such that $\sum_{i=1}^m w_i = 1$, $w_i \geq 0$, $\left| \frac{1}{n_1} \sum_{i:D_i=1} X_{ik} - \sum_{i=1}^m w_i X_{ik} \right| \leq \delta_k$

- Minimum variance weights that approximately balance covariates.
 - Amount of allowed imbalance, δ_k , selected by researcher.
 - Can include X_{ik}^2 etc to balance other parts of the distribution.

Have a great weekend! :)

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